

Reissner-Nordstrom and charged gas spheres.

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ABSTRACT. The main point of this paper is a suggestion about the proper treatment of the photon gas in a theory of stellar structure and other plasmas. This problem arises in the study of polytropic gas spheres, where we have already introduced some innovations. The main idea, already advanced in the context of neutral, homogeneous, polytropic stellar models, is to base the theory firmly on a variational principle. Another essential novelty is to let mass distribution extend to infinity, the boundary between bulk and atmosphere being defined by an abrupt change in the polytropic index, triggered by the density. The logical next step in this program is to include the effect of radiation, which is a very significant complication since a full treatment would have to include an account of ionization, thus fields representing electrons, ions, photons, gravitons and neutral atoms as well. In way of preparation, we consider models that are charged but homogeneous, involving only gravity, electromagnetism and a single scalar field that represents both the mass and the electric charge; in short, a non-neutral plasma. While this work only represents a stage in the development of a theory of stars, without direct application to physical systems, it does shed some light on the meaning of the Reissner-Nordstrom solution of the modified Einstein-Maxwell equations., with an application to a simple system.

1. Introduction

A fundamental field theory is characterized by the fact that the number of degrees of freedom is specified from the start, usually revealed by counting the independent field components that appear in the action. The standard approach to stellar structure is very different. First, there is the mysterious vector field that is said to represent the flow; it is usually normalized to unity, a step that we find very hard to accept. In the study of spherically symmetric, equilibrium configurations it contributes at most one degree of freedom, in evolution, up to three, depending on the symmetry. Next there are two scalar fields identified as energy (or mass) density and pressure density. Additional fields are

introduced as needed, including additional densities, pressures, luminosity. Typically, the introduction of each new field is accompanied by one or more additional equations that help to pin it down. (Surprisingly, the density of the photon gas is not dignified by the introduction of an independent degree of freedom.) This type of approach requires a tremendous amount of physical intuition, far exceeding the understanding of this author. And he misses the guidance that comes with an action principle.

A preliminary study of homogeneous, polytropic models of ideal stars [F1] has shown that the incorporation of an action principle leads to a more coherent theory. The mass was shown to be a constant of the motion and related to a conserved current. The study was limited to irrotational flows, and this limitation will be retained in the present paper. The action includes, besides the Hilbert action for the metric field, a straightforward, relativistic generalization of the standard hydrodynamic action for irrotational motion [FW], namely

$$A_{Matter} \propto \int dx \sqrt{-g} \left(\frac{\rho}{2} (g^{\mu\nu} \psi_{,\mu} \psi_{,\nu} - c^2) - V[\rho] \right). \quad (1.1)$$

It involves a single scalar field ρ and the velocity potential ψ with the dimension of length. A more realistic treatment of stars such as the sun must take radiation into account. The next logical step is therefore to invent an action principle that involves charged matter as well as electromagnetic and gravitational fields. To be realistic, one would have to include free electrons, ions, and atoms. Here we shall deal with a simpler situation, not applicable to the sun, but perhaps to certain layers of some stellar atmospheres, and other non neutral plasmas.

It may be possible to account for the effect of radiation by simply adding the radiation pressure to the pressure of the matter component, as may seem to be justified by the fact that the value taken by the Lagrangian density on shell can be interpreted as pressure. The phenomenological device of taking them to be proportional to each other [E][C] may also be justified. It is expected that the theory will enlighten us on this point.

The simplest expansion of the model to include electromagnetism utilizes the fact that it has a conserved current,

$$J^\mu = \sqrt{-g} \rho g^{\mu\nu} \psi_{,\nu}. \quad (1.2)$$

We shall proceed by introducing the minimal electromagnetic interaction associated with this current. In addition, we could just add the Maxwell action,

$$A_{Maxwell} = \frac{-1}{16\pi} \int dx \sqrt{-g} g^{\mu\nu} g^{\lambda\rho} F_{\mu\lambda} F_{\nu\rho}.$$

But this is probably not the right thing to do, as we shall try to show.

The simplest of all couplings of the metric to matter would use the action for a neutral scalar field, as (1.1) but with $\rho = 1$ and $V = 0$. This is wholly inadequate for dealing with a continuous matter distribution described by macroscopic density and pressure. To generate a pressure one has to include interactions, and as it turns out the inclusion of the density ρ and the potential $V[\rho]$ is an effective way to do that. The Maxwell action is inappropriate for the same reason; we wish to include the effect of radiation, an incoherent superposition

of a large number of photons, endowed as the matter distribution with a pressure and a density of its own. The traditional approach is to add the radiation pressure, by hand, to the pressure of matter, without attempting to include it as an additional variable in the dynamical scheme. What appears to be needed is a contribution to the action of the form

$$A_{\text{Radiation}} = \int dx \sqrt{-g} \left(\frac{-\sigma}{16\pi} g^{\mu\nu} g^{\lambda\rho} F_{\mu\lambda} F_{\nu\rho} - W[\sigma] \right). \quad (1.3)$$

Here σ is a dimensionless scalar field and $W[\sigma]$ is an internal energy density. Here too, it is expected that the introduction of a density and a potential is an effective way to introduce the interactions that give rise to a photon pressure.

Inevitably, this leads to the idea that the gravitational radiation that must be present has to be treated analogously, but that will not be developed in this paper. (See [MM].)

This study will be further limited to systems with spherical symmetry and to potentials of the form $V[\rho] \propto \rho^\gamma$, leading to a polytropic equation of state.

(Added in response to referee. To be inserted here. See response to referee.)

The idea that the Reissner-Nordstrom metric can be interpreted in terms of a charged gas was already advanced by Felipe *et al* [dF1][dF2].

(End of addition.)

Outline

Section 2 is a somewhat slowly paced introduction to explain our point of view. Section 3 is a study of static configurations of charged polytropes, with an investigation of what we take to be the correct physical interpretation of the Reissner-Nordstrom metric. Among the equations of motion we emphasize (1) the conservation law for the current, (2) variation of the mass density ρ gives a relation between potentials that, in the limit when ρ is zero, leads directly to the extremal Reissner-Nordstrom metric, and (3) variation with respect to the photon density σ gives an expression for σ in terms of the field strength.

To complete the definition of the model we take

$$V[\rho] = a\rho^\gamma,$$

with a and γ piecewise constant. This gives rise to an polytropic equation of state with index n ,

$$p = \frac{a}{n} \rho^\gamma, \quad \gamma = 1 + \frac{1}{n}.$$

It is well known that a value $n < 5$ in the bulk of the star is compatible with regularity at the center. At great distances it is customary to take $\rho = 0$, certainly an excellent approximation, but another polytrope, with $n > 5$ is also possible. The boundary presents difficult problems in either case, a difficulty that we circumvent by having the change in

the index triggered by the density. Boundary conditions are imposed only at the center and at infinity and all the fields are continuous at the ‘boundary’. The photon internal energy $W[\sigma]$ is chosen so as to give the usual equation of state, $p_{\text{ph}} = T_0^0/3$ for the photon pressure p_{ph} .

The theory has an interesting connection to the Reissner-Nordstrom solution of Einstein’s field equations for empty space, and the extremal solution (where the electrostatic repulsion exactly cancels the gravitational attraction) plays a role.

Direct applications to real physical systems are not discussed. The main result of numerical calculations is that the photon density profile may be very different from the matter distribution. If this carries over to the case of neutral plasmas then it will instill some skepticism with respect to the assumptions that are usually made about the radiation pressure.

A discussion of lessons learned from this investigation is deferred to the last section.

2. Background on method

2.1. Geodesics

A very attractive feature of General Relativity is the fact that a test particle in a given metric field moves along a geodesic, minimizing the action

$$A_{Particle} \propto \int_{\gamma} ds, \quad ds^2 = g_{\mu\nu} dx^{\mu} dx^{\nu}.$$

It is a well defined line integral along a path γ with fixed endpoints; it is independent of any parameterization of γ . If $\tau : \gamma \rightarrow \mathbb{R}$ is a parameterization, then

$$A_{Particle} \propto \int d\tau \sqrt{g_{\mu\nu} U^{\mu} U^{\nu}}, \quad U^{\mu} = dx^{\mu}/d\tau.$$

The velocity of the particle at any point on the path is

$$\vec{v} = d\vec{x}/dt, \quad d\vec{x} := (dx^1, dx^2, dx^3);$$

it has a direct physical meaning, while the 4-velocity field U does not. Since $\vec{v} = \vec{U}/U^0$, the vector field U is defined up to a scale transformation. The scalar field $g_{\mu\nu} U^{\mu} U^{\nu}$ is a constant of the motion and that allows fixing its value along the path, therefore the scale may be fixed by requiring, for example, by setting $g_{\mu\nu} U^{\mu} U^{\nu} = 1$. By referring to a “test particle” one implies that the reaction of the metric field to the presence of the particle is being neglected. In the case of several test particles one is also neglecting any interaction between the particles; in this case each particle has its own path, its own 4-velocity field and its own “proper time”. It is possible to fix the fourth, unphysical component (equivalently, the scale) of each, by setting $g_{\mu\nu} U_1^{\mu} U_1^{\nu} = g_{\mu\nu} U_2^{\mu} U_2^{\nu} = \dots = 1$, but as the number of particles grows the neglect of any reaction on the field, and of any interaction between the particles, makes such conditions increasingly implausible.

2.1. Field equations

Einstein’s field equations deal, in the first instance, with the determination of the metric field produced by a given matter distribution,

$$G_{\mu\nu} = \frac{8\pi G}{c^2} T_{\mu\nu}, \tag{2.1}$$

where $G_{\mu\nu} = R_{\mu\nu} - (1/2)g_{\mu\nu}R$ and T is the energy momentum tensor of the matter distribution. All the familiar, elementary, covariant field theories possess energy momentum tensors with the requisite properties, most important of which is that of being covariantly conserved,

$$T_{\mu;\nu}^{\nu} = 0, \tag{2.2}$$

as it must be because of the (contracted) Bianchi identities that are satisfied identically by the Einstein tensor. It is a direct consequence of the fact that these theories are characterized by an invariant action principle, with some action A and

$$T_{\mu\nu} = 2 \frac{\delta A}{\delta g^{\mu\nu}}. \tag{2.3}$$

In general, the right hand side of Eq.(2.1) is supposed to represent ‘matter’, the source of the metric field. It is plausible, and it seems almost mandatory, that it have the form (2.3). But in studies of stellar structure ‘matter’ is extremely complicated and explicit, realistic expressions for the action, in terms of elementary particle fields, is not to be dreamt of.

It is necessary to introduce statistical or more precisely thermodynamical variables. If one excludes crystalline structures, then it is natural to follow Tolman [T], who assumed that T have the representation

$$T_{\mu\nu} = c^{-2}(\hat{\rho} + \hat{p})U_\mu U_\nu - \hat{p}g_{\mu\nu}, \quad (2.4)$$

where $\hat{\rho}$ and \hat{p} are scalar fields and U is a 4-vector field that is associated with the local velocity of flow,

$$\vec{v}(x) = \vec{U}(x)/U^0(x). \quad (2.5)$$

While the 4-velocity of a test particle is defined over the path, this one is defined over the entire space time manifold, in accord with the interpretation of the continuum as a limit of a dense distribution of particles. Here too one has the problem of interpreting the fourth component, and the remedy has always been to postulate the “normalization”

$$g(U) := g_{\mu\nu}(x)U^\mu(x)U^\nu(x) = 1; \quad (2.6)$$

that is, the highly implausible continuum limit of the normalization used for n non-interacting particles.

In defense of (2.6) it is true that any other normalization, $g(U) = f^2$, say, with f a scalar field, can be reduced to (2.6) by a rescaling of $U, \hat{\rho}$ and \hat{p} . This would affect the physical interpretation of $\hat{\rho}$ and \hat{p} , and hence the equation of state. In the case of slow motion and weak gravitational fields Eq.(2.6) reduces to $U^0(x) = 1$, and in this case $\hat{\rho}$ and \hat{p} are interpreted as energy density and pressure, respectively, to be treated as standard thermodynamic variables. But the subsequent extrapolation of thermodynamics to velocities approaching that of light, and strong gravitational fields, is a bold extension of the equivalence principle, making Eq.(2.6) an *ad hoc* assumption with real physical consequences.

A more serious result of adopting Eq.(2.6) is that it cannot be reconciled with an action principle. The required covariant conservation law (2.2) thus becomes an axiom in itself, without an action principle to support it. The power of action principles in the formulation of physical theories is almost universally recognized, so one is not surprised to find that attempts have been made to find an action consistent with (2.6) [S][T], but so far these efforts have been formalistic and without real content, which is why they have had no influence on the development of the subject.

In this paper we are beginning to work with a several kinds of “matter”, each making a contribution to the total action; in this case the formula (2.4) can no longer hold and the question of normalization becomes moot. In the case of a static configuration it is still plausible to identify T_0^0 with the total energy density.

2.3. The Newtonian limit

The equations of static, Newtonian gravity, a limiting case of Einstein's theory, are Poisson's equation,

$$\Delta\phi = 4\pi G\rho, \quad (2.7)$$

where ρ is the mass density and ϕ is the gravitational potential, and the hydrostatic condition

$$\rho\vec{\nabla}\phi + \vec{\nabla}p = 0. \quad (2.8)$$

An important theory of irrotational hydrodynamics is based on Bernoulli's equation. This theory can be formulated as a variational problem with the action [FW]

$$\int d^3x dt \left(\rho(\dot{\Phi} - \vec{v}^2/2 - \phi) - V \right). \quad (2.9)$$

Here Φ is the velocity potential, $\vec{v} = -\vec{\nabla}\Phi$, V is the internal energy and ϕ is the external gravitational potential. In the isentropic case $V = V[\rho]$ is determined by ρ and the pressure is

$$p = \rho \frac{dV}{d\rho} - V. \quad (2.10)$$

This coincides (on-shell) with the action density, which explains why pressure is additive. In this case the equations that govern equilibrium configurations are Poisson's equation for ϕ , the continuity equation for the current $\rho\vec{v}$, and the variational equation

$$\dot{\Phi} = \vec{v}^2/2 + \phi + \frac{dV}{d\rho}. \quad (2.11)$$

Taking the gradient of this equation one recovers, with the help of (2.10), the hydrostatic equation (2.8).

An effect of specifying an action is thus to replace the differential, hydrostatic condition by an integrated form of it, with a fixed choice of the integration constant. In Newtonian gravity the zero point of the potential has no meaning, so that this fixing of an integration constant is irrelevant, but it has important consequences in General Relativity.

2.4. Irrotational flow in General Relativity

A straightforward generalization of the action (2.9) is

$$A_{Matter} = \rho_{cr} \int d^4x \sqrt{-g} \left(\frac{\rho}{2} (g^{\mu\nu} \psi_{,\mu} \psi_{,\nu} - c^2) - V[\rho] \right), \quad (2.12)$$

where ρ and ψ are scalar fields. The field ρ is dimensionless; the correct density dimension is provided by the factor ρ_{cr} . The non-relativistic theory is recovered by setting

$$\dot{\psi} = c^2 + \dot{\Phi}, \quad \partial_i \psi = \partial_i \Phi, \quad i = 1, 2, 3, \quad (2.13)$$

or simply $\psi = c^2 t + \Phi$.

We consider the theory to be defined by the action. The possibility of a thermodynamical interpretation, for some choice of the functional $V[\rho]$, is expected, but it is not an axiom. This is an important departure from the traditional approach.

It is natural to define the material pressure as before, by Eq.(2.10), $p = \rho(dV/d\rho) - V$, a scalar field that, on the trajectory, coincides with the Lagrangian density.

Variation of the independent fields ψ and ρ leads to the equation of continuity

$$\partial_\mu J^\mu = 0, \quad J^\mu := \rho \sqrt{-g} g^{\mu\nu} \psi_{,\nu} \quad (2.14)$$

and the analogue of (2.11), namely

$$\frac{1}{2}(g^{\mu\nu} \psi_{,\mu} \psi_{,\nu} - c^2) = \frac{dV}{d\rho}. \quad (2.15)$$

The energy momentum tensor associated with the action (2.12) is given on shell by

$$T_{\mu\nu} = \rho \psi_{,\mu} \psi_{,\nu} - p g_{\mu\nu}. \quad (2.16)$$

It is of the Tolman form, except that the vector field U is here restricted to be of the gradient type. Eq.s (2.14-15) imply that $T_{\mu;\nu}^\nu = 0$. On the other hand, to prove it, one needs only the gradient of (2.15), namely

$$\partial_\lambda p = \frac{\rho}{2} \partial_\lambda (g^{\mu\nu} \psi_{,\mu} \psi_{,\nu}). \quad (2.18)$$

Thus one sees that here too, as in the non-relativistic theory, the first effect of introducing an action is to fix an integration constant.

2.5. Implications of variational formulation for boundary conditions

In any analysis of stellar structure the static solutions play a dominant role. They describe the equilibria and, in addition, time development is often adiabatic and hence a succession of equilibrium configurations. It is important to notice that, in Tolman's theory, some of the metric fields are represented only by their derivatives. We consider the case of static configurations in a metric that embodies spherical symmetry and, in a set of coordinates t, r, θ, ϕ , takes the form

$$ds^2 = e^\nu dt^2 - e^\lambda dr^2 - r^2 d\Omega^2, \quad (2.19)$$

with ν, λ depending on r only. The field λ appears in the standard field equations (Tolman's approach), as does $\nu' := d\nu/dr$, but ν itself does not. The equations are therefore insensitive to a constant shift of ν , for this reason the boundary conditions are less restrictive than in the model based on the action principle.

An external, Schwarzschild metric has $\lambda + \nu = 0$; therefore, at the surface of a star, where $r = R$, say, the following boundary condition must hold,

$$\lambda(R) + \nu(R) = 0. \quad (2.20)$$

Consider a homogeneous, polytropic model of the sun:

$$V = a\rho^{4/3}, \quad a \text{ constant.}$$

A determination of the best value of the constant a would take us too far, see [F3], so we eliminate it,

$$\frac{dV}{d\rho} = 4\frac{p}{\rho}, \quad (2.21)$$

then use the gas law to define the temperature,

$$T = \frac{\mu}{\mathcal{R}} \frac{p}{\rho}. \quad (2.22)$$

In the static case $\dot{\psi} = 1$, $\vec{\nabla}\psi = 0$ and Eq.(2.15) becomes

$$-\phi = \frac{1}{2}(e^{-\nu} - 1) = 4\frac{p}{\rho} = 4\frac{\mu}{\mathcal{R}}T. \quad (2.23)$$

If ρ vanishes for $r > R$ then matching the metric to the external Schwarzschild metric at $r = R$ gives $\phi(R) = -GM/R$, where M is the mass. Thus finally,

$$T(R) = \frac{1}{4} \frac{\mu}{\mathcal{R}} \frac{GM}{R}.$$

The standard approach gives this temperature distribution (2.23) *modulo* an additive constant; matching to the external Schwarzschild metric is inconsequential. In this paper we shall discover additional confirmation of the relevance of the stronger condition (2.15).

2.6. Stability, mass and improved boundary conditions

To complete the matter model we take

$$V[\rho] = a\rho^\gamma,$$

with a and γ piecewise constant. This gives rise to the isentropic equation of state,

$$p = \frac{a}{n} \rho^\gamma, \quad n = \frac{1}{\gamma - 1},$$

of a polytrope with index n . It is well known that taking $n < 5$ in the bulk of the star ensures that the density becomes zero at some finite radius..

We have reported calculations of the static solutions of this model, for various values of the polytropic index [F3]. Rather than follow the traditional method of defining the boundary of the star to be at the first zero of the density, we matched the interior solution to an exterior Schwarzschild metric. This strategy does not work in the traditional context, but the stronger wave equations obtained from variational principle determine the radius uniquely as the place where the condition (2.20) holds, the only place where matching to the exterior metric is possible.

Subsequently we studied the stability of these static configurations [F3]. It turned out that we could not find sufficient guidance to choose between the possible boundary conditions. In particular, it seemed strange that the mass, defined asymptotically by the exterior metric, turned out to echo the oscillations of the star. As a first attempt to remedy the situation we replaced the exterior Schwarzschild metric by another polytrope, with $n > 5$. Unfortunately, that was not enough to settle the question of appropriate boundary conditions. It was decided to give up the idea of a fixed boundary altogether. After all, the ‘boundary’ of a double polytrope is just a place where the polytropic index changes, more or less abruptly. If the star starts from a diffuse, gaseous state, then this change must come about as a result of the increase in density that follows from the gradual gravitational collapse and in response to the threat of a singularity developing near the center.

The change in the index at “the boundary” is evidently a result of the increase in density, and all uncertainty concerning the correct choice of boundary conditions can be avoided by making the interdependence of index and density explicit, posing for example,

$$n = 3 + \frac{3}{1 + \rho^K},$$

where $K \gg 1$ and $\rho = 1$ is a critical density. The boundary conditions at the center are supplemented by the natural requirement that the fields decrease at infinity. As it turns out, this radical change in the treatment of the boundary has very little effect on the static solutions, but a very salutary influence on stability. Boundary conditions are imposed at the center and at infinity; all static solutions found appear to be quite stable to radial excitations. In addition, it turned out that the new boundary conditions (at infinity) ensure that the mass, defined by the asymptotic metric, but also related to the space integral of the charge density ρ (with the correct measure!) is a constant of the motion. This is regarded as an important advantage over the traditional theory. The same strategy is followed in the present paper.

3. Radiation and charged matter

3.1. The model, main features

The simplest form of charged matter is a distribution that consists of one type of charged particles; in this case the charge density is just $e\rho$, where e is a unit of charge. To introduce the radiation field we first include the radiation action

$$A_{\text{Radiation}} = \int dx \sqrt{-g} \left(\frac{-1 - \sigma}{16\pi} g^{\mu\nu} g^{\lambda\rho} F_{\mu\lambda} F_{\nu\rho} - W[\sigma] \right) =: \int dx \sqrt{-g} \mathcal{L}_{\text{rad}}. \quad (3.1)$$

Some justification for this expression, that differs from the Maxwell action by the inclusion of a “photon density” field σ and an internal photon energy $W[\sigma]$, was offered in the Introduction. For the functional $W[\sigma]$ a preliminary suggestion may be to take

$$W[\sigma] \propto \sigma^2;$$

this choice will be motivated below, with other possibilities, and critically examined in Section 5. Note that the field strength represents the global, coherent electromagnetic field; the photon gas appears only by way of the density σ . We defer discussion of the physical or thermodynamic meaning of this field.

In addition we include a coupling of the potential to the conserved current. The following modified matter action is formally gauge invariant,

$$A_{Matter} = \int d^4x \sqrt{-g} \left(\frac{\rho}{2} (g^{\mu\nu} \psi_{;\mu} \psi_{;\nu} - 1) - V[\rho] \right), \quad \psi_{;\mu} = \partial_\mu \psi + e A_\mu. \quad (3.2)$$

The constant e is a unit of charge. For the potential $V[\rho]$ we shall adopt a simple form that leads to an isentropic equation of state; see below.

In the non-relativistic limit, defined by setting $\psi = ct + \Phi$, $\vec{v} = -\vec{\nabla} \Phi$,

$$A_{Matter} = \rho_{cr} \int d^4x \left(\rho (\dot{\Phi} + e A_0 - \vec{\pi}^2/2) - V[\rho] \right), \quad \vec{\pi} = \vec{v} - e \vec{A}.$$

Wave equations, besides the modified Maxwell equations, are

$$\dot{\Phi} + e A_0 - \vec{\pi}^2/2 = dV[\rho]/d\rho, \quad \dot{\rho} + \vec{\nabla} \cdot (\rho \vec{\pi}) = 0$$

As usual, define the matter pressure p to be the Lagrangian density or, equivalently, by the familiar formula

$$p = \rho \frac{dV}{d\rho} - V. \quad (3.3)$$

The continuity equation shows that $\vec{\pi}$ must be interpreted as the velocity. The force balance equation should be related to the gradient of the first wave equation,

$$-(d/dt)v_i + e \partial_i A_0 - \vec{\pi} \cdot \partial_i \vec{\pi} = \partial_i \frac{dV}{d\rho} = \frac{1}{\rho} \partial_i p,$$

or

$$\frac{d}{dt} \pi_i + \vec{\pi} \cdot \partial_i \vec{\pi} + \frac{\partial_i p}{\rho} = e \left(F_{i0} + (\vec{\pi} \wedge \vec{B})_i \right). \quad (3.4)$$

This agrees with standard theory of non-neutral plasmas [D] if we identify the velocity with $\vec{\pi}$ and the stress tensor with δp ($= \text{diag } p$). We have not found any discussion of a the possible presence of a photon gas in the literature on nonneutral plasmas.

The energy momentum tensor has the two contributions, from matter and radiation:

$$T_{\mu\nu} = \rho \psi_{;\mu} \psi_{;\nu} - g_{\mu\nu} p + \frac{-1 - \sigma}{4\pi} g^{\lambda\rho} F_{\mu\lambda} F_{\nu\rho} - g_{\mu\nu} \tilde{\mathcal{L}}. \quad (3.5)$$

3.2. The conserved current and the mass

The conservation law (2.14) can be integrated to yield

$$\frac{d}{dt} \int_0^\infty \sqrt{e^{(-\nu+\lambda)/2}} r^2 \rho(\dot{\psi} + eA) dr = \left[\sqrt{e^{(\nu-\lambda)/2}} r^2 \rho(\psi' + eA_1) \right]_0^\infty.$$

In view of the boundary conditions at the origin,

$$\frac{d}{dt} \int_0^\infty \sqrt{e^{(-\nu+\lambda)/2}} \rho(\dot{\psi} + eA_0) r^2 dr = \lim_{r \rightarrow \infty} \left[\sqrt{e^{(\nu-\lambda)/2}} r^2 \rho(\psi' + eA_r) \right].$$

The factor ρ on the right hand side suggests that there is no flux at infinity, but in fact the flux $r\rho(\psi' + A_1)$ is equal to $-\dot{\lambda}/8\pi G$ by virtue of Einstein's equations. For a static configuration both sides of the equation are zero; for a first order deviation from a static configuration we have

$$\frac{d}{dt} \int_0^\infty \sqrt{e^{(-\nu+\lambda)/2}} r^2 \rho \delta(\dot{\psi} + eA_0) dr d\Omega = \frac{1}{2G} \lim_{r \rightarrow \infty} (r \delta \dot{\lambda}).$$

If the perturbed and unperturbed metrics both tend to Schwarzschild at infinity, then $r\dot{\lambda} \rightarrow 2\dot{m}G$ and

$$\frac{d}{dt} \int_0^R \sqrt{e^{(-\nu+\lambda)/2}} r^2 \rho (\delta\dot{\psi} + e\delta A_0) dr d\Omega = \dot{m}.$$

It is not *a priori* obvious that the left side is a constant of the motion, but the result of our calculations is that $r\delta\dot{\lambda}$ and $r\delta A_0$ tend to zero at infinity so that in fact $\dot{m} = \dot{q} = 0$. The integral

$$\int_0^\infty \sqrt{-g} g^{tt} \rho r^2 (1 + eA_0) dr d\Omega$$

is a constant of the motion.

3.3. Static configurations

Since $\vec{\pi}$ is interpreted as velocity, we may call 'static' a configuration in which $\vec{\pi} = 0$, $\dot{\psi} = 1$ and $\dot{\rho} = \dot{A}_0 = \dot{\nu} = \dot{\lambda} = 0$. There is a gauge in which $\vec{A} = 0$, Maxwell's equations reduce to Poisson's equation. With $A_0 \rightarrow A$,

$$(e^{-(\nu+\lambda)/2} r^2 (1 + \sigma) A')' = 4\pi \rho e^{(\lambda-\nu)/2} r^2 (1 + eA), \quad (3.6)$$

and variation of the field σ gives

$$\frac{-1}{8\pi} g^{tt} g^{rr} F_{tr}^2 = \frac{1}{8\pi} e^{-\nu-\lambda} A'^2 = \frac{dW}{d\sigma}. \quad (3.7)$$

Derivation of (3.7) with respect to the radial coordinate gives

$$\frac{1}{8\pi} e^{-\nu-\lambda} \left(-(\nu + \lambda)' A'^2 + 2A' A'' \right) = \frac{P'_{\text{ph}}}{\sigma}, \quad (3.8)$$

where

$$p_{\text{ph}} := \sigma \frac{dW}{d\sigma} - W$$

may be interpreted as the pressure of the photon gas. The first term on the left hand side is negative, which favors a pressure that decreases with distance, just as is the case with ordinary matter. If A'^2 decreases with distance then the second term is also negative, which represent an additional attractive force holding the photons together. At a large distance this term predominates.

Of the matter wave equations there remains only the integrated hydrostatic condition,

$$g^{00}(1 + eA)^2 - 1 = 2 \frac{dV}{d\rho} = 2a\gamma f. \quad (3.9)$$

The Emden function f is related to the density and the pressure by

$$\rho = f^n, \quad p = \frac{a}{n} f^{n+1}, \quad (3.10)$$

where a is a constant that depends on the type of matter and that shall be regarded as a free parameter. Derivation of (3.9) with respect to the radius gives

$$e^{-\nu} \left(-\nu'(1 + eA)^2 + 2eA'(1 + eA) \right) = \frac{p'}{\rho}.$$

The first term is negative as usual. The second term is due to the electrostatic interaction; since it is evidently repulsive we conclude that eA' must be positive at large distances.

We shall look for static solutions with spherical symmetry, with the metric (2.19),

$$ds^2 = e^\nu dt^2 - e^\lambda dr^2 - r^2 d\Omega^2,$$

the coefficients now depending on the coordinate r only, $0 < r < \infty$. Einstein's equations reduce to

$$\begin{aligned} 1 - (re^{-\lambda})' &= 8\pi Gr^2 \left(e^{-\nu} \rho (1 + eA)^2 - p \right) + 8\pi Gr^2 \left(e^{-\nu-\lambda} \frac{1+\sigma}{4\pi} A'^2 - \mathcal{L}_{\text{rad}} \right), \\ \nu' + \lambda' &= 8\pi G \rho r e^{-\nu} (1 + eA)^2. \end{aligned} \quad (3.11)$$

They must be supplemented with (3.7) and (3.9).

3.4. Reissner-Nordstrom and infinite distributions

The exact Reissner-Nordstrom solution of Einstein's equations in empty space has two free parameters, the mass m and the charge q ,

$$e^\nu = e^{-\lambda} = 1 - 2mG/r + q^2 G/r^2, \quad A = -q/r. \quad (3.12)$$

In the case that $\rho = 0$ our equation (3.9) gives $g_{00} = (1 + eA)^2$ which agrees with (3.12) if $q^2 = m^2 G$ and $e^2 = G$; that is, in the case of the extremal Reissner-Nordstrom solution.

In this special case the radiation pressure is balanced by the attractive force mentioned after Eq.(3.8).

If $q^2 > m^2 G$ the electrostatic repulsion dominates and it is to be expected that the only static solution is a space that is empty of matter except possibly near the center, where gravity may become strong. In this case, when there is no horizon, we agree with the traditional interpretation of the Reissner-Nordstrom metric as the metric of a space that is empty except for a small region near the center. If instead $q^2 < m^2 G$ we expect to find solutions with non zero matter density. The metric may approach that of Reissner-Nordstrom at great distances but it will have no horizon. The exact Reissner-Nordstrom solution with $\rho = 0$ outside the horizon can perhaps be interpreted as a singular limit of a sequence of space times with mass concentrated near and inside the horizon.

In general we expect that, if the density is integrable at infinity, there are solutions that behave for large r , to leading order in $1/r$, as the Reissner-Nordstrom solution. Assuming that there are numbers m, q and ϵ such that, asymptotically,

$$\lambda - 1 \sim 1 - \nu \sim \frac{2m}{r}, \quad A \sim -\frac{q}{r}, \quad \rho \sim r^{-\epsilon}.$$

we find from (3.10) that $\epsilon \geq n$ and that

- If $\epsilon > n$, then $eq = mG$.
- If $\epsilon = n$, then $mG - eq = (2a/n)\gamma \lim rf$, so that $eq < mG$.

There are no solutions of this kind if $eq > mG$. There are no solutions such that that $\rho = 0$ everywhere. Further analysis of the asymptotics indicate that $\epsilon = n$ so that $f \propto 1/r$.

3.5. The photon equation of state

Asymptotically, at large distances, the equations imply that the density σ is of the order of $1/r^4$. Both W and W' fall off as $1/r^4$, this gives internal evidence about the functional $W[\sigma]$. Eq.(3.7) implies that

$$W = b \sigma^2 (1 + b_1 \sigma + \dots),$$

with b, b_1, \dots constant. This gives the pressure of the photon gas,

$$p_{\text{ph}} = \sigma W' - W = b \sigma^2 (1 + 2b_1 \sigma + 3b_2 \sigma^2 \dots).$$

The contribution of the photon gas to the “energy density” T_t^t is

$$\frac{-\sigma}{8\pi} F^2 - p_{\text{ph}} = 2\sigma W' - \sigma W' + W = \sigma W' + W.$$

If we retain only the first term in the expression for $W[\sigma]$ it is

$$\rho W' + W = 3b \sigma^2 = 3p_{\text{ph}},$$

which is the familiar equation of state for a photon gas. The most reasonable equation of state, both from contextual and thermodynamic considerations is thus associated with the simple expression

$$W = b \sigma^2, \tag{3.13}$$

where b , unlike a , is a fundamental constant, the value unknown to us so far.

Further discussion of this choice of the functional $W[\sigma]$ is deferred to the last section.

3.6. Exact solutions

It is curious that the matterless Reissner-Nordstrom metric can be derived from (3.9), the equation derived from the action principle by variation of ρ , in the special case that $\rho = 0$. The complete set of equations is in this case, when $\rho = 0$,

$$\begin{aligned}\nu + \lambda &= 0, \\ r^2(1 + \sigma)A' &= q^2 = \text{constant}, \\ A'^2 &= 16\pi b\sigma, \\ 1 - (re^{-\lambda})' &= 8\pi Gr^2(2b\sigma + 3b\sigma^2),\end{aligned}$$

and, from variation of ρ ,

$$e^\nu = (1 + eA)^2.$$

The first 4 equations give us

$$(1 + \sigma)^2\sigma = (q^4/16\pi b)r^{-4},$$

and if $A' = \sqrt{64\pi b/3} X$,

$$(3 + 4X^2)X = \sqrt{\frac{27}{64\pi b}} \frac{q^2}{r^2} = c/r^2.$$

If θ is defined by $X = \sinh \theta$, then the left side is $\sinh 3\theta$ and an explicit formula for A' is obtained in the form

$$A' = \sqrt{\frac{64\pi b}{3}} \sinh\left(\frac{1}{3} \sinh^{-1} \frac{c}{r^2}\right).$$

The metric is determined and when it is eliminated we are left with the following,

$$1 - \left(r(1 + eA)^2\right)' = 8\pi Gr^2(\sigma W' + W + W').$$

The existence of a solution with no matter thus implies a special form of the functional $W[\sigma]$. This solution gives A' and σ as smooth, positive functions that fall off as $1/r^2$ and $1/r^4$, respectively. Near the center,

$$A \propto r^{1/3}, \quad \sigma \propto r^{-4/3}, \quad \lambda \propto r^{2/3}.$$

This is not what is considered normal on physical grounds. The conclusion is that the Reissner-Nordstrom solution is not rendered more physical by the inclusion of a photon gas. Changing the equation of state does not help. It remains to be seen whether the inclusion of matter leads to more reasonable results. For this we must appeal to numerical solutions.

4. Numerical calculations

The full set of equation in the static case consists of Einstein's equations,

$$\begin{aligned} e^{-\lambda}(r\lambda' - 1) + 1 = & 8\pi(G\rho_{cr})r^2\left(e^{-\nu}f^n(1 + eA)^2 - (a/n)f^{n+1}\right) \\ & + 8\pi Gr^2(2b\sigma + 3b\sigma^2), \end{aligned} \quad (4.1)$$

$$\nu' + \lambda' = 8\pi(G\rho_{cr})re^{-\nu}\rho(1 + eA)^2, \quad (4.2)$$

Maxwell's equations,

$$(e^{-(\nu+\lambda)/2}r^2(1 + \sigma)eA')' = 4\pi(e^2\rho_{cr})r^2\rho e^{(\lambda-\nu)/2}(1 + eA), \quad (4.3)$$

and the equation that come from variation of the densities,

$$\frac{1}{8\pi}e^{-\nu-\lambda}A'^2 = W', \quad e^{-\nu}(1 + eA)^2 - 1 = 2a\gamma f. \quad (4.4 - 5)$$

We set $b = \rho_{cr}\beta$, absorb e into A and set $\alpha = e^2/G$. Replacing $(G\rho_{cr})$ by 1 in the equations amounts to fixing the unit of length. The final form of the equations is

$$\begin{aligned} e^{-\lambda}(r\lambda' - 1) + 1 = & 8\pi r^2\left(e^{-\nu}f^n(1 + A)^2 - (a/n)f^{n+1}\right) \\ & + 8\pi r^2(2\beta\sigma + 3\beta\sigma^2), \end{aligned} \quad (4.1)$$

$$\nu' + \lambda' = 8\pi re^{-\nu}\rho(1 + A)^2, \quad (4.2)$$

$$(e^{-(\nu+\lambda)/2}r^2(1 + \sigma)A')' = 4\pi\alpha r^2\rho e^{(\lambda-\nu)/2}(1 + A), \quad (4.3)$$

$$\frac{1}{8\pi}e^{-\nu-\lambda}A'^2 = 2\alpha\beta\sigma, \quad e^{-\nu}(1 + A)^2 - 1 = 2a\gamma f, \quad (4.4 - 5)$$

with $\alpha = e^2/G$.

As always, we assume regularity of all fields at the center. We start from $\alpha = \beta = 0$. In this case the electromagnetic sector makes no contribution and we already have solutions for $(n_1, n_2) = (3, 6)$, $a > 6$, given in the Table. We choose a value of a at random, $a = 1/10$, and begin to increase the values of α and β . All the parameters are now fixed and we adjust the values of $\nu(0)$ and $A(0)$, with $A'(0) = \lambda(0) = 0$, until we get a solution for the function ν such that $2m = \lim r\nu$ at infinity is finite. The idea is to reach the extremal value $\alpha = e^2/G = 1$, adjusting β and a as necessary as we gradually increase α from zero. The value of the parameter β turns out to be nearly irrelevant. More precisely, a rescaling of β does nothing more than rescale the solution for the field b by the inverse factor.

The search of solutions that exist for a discrete set of points in a 2-dimensional parameter space is very laborious. Representative solutions are given in the Table. The highest value that was attained for the Reissner-Nordstrom parameter $\alpha = e^2/G$ was .65, but this is probably not an absolute limit.

Table								
$a = \frac{1}{6}$	$b = 1$	$-\nu(0)$	$f(0)$	$-A(0)$	R	$2mG$	q	maximum
	$k = 0$.5670415	1.749		.0962	.03045		
	$k = .15$.61132	1.75	.03221	.10	.03935	.00256	$\sigma(.10) = .0052$
	$k = .40$.699495	1.42	.0975216	.115	.0654	.00995	$\sigma(.13) = .0123$
$a = \frac{1}{10}$	$k = .45$							
	$b = .10$.448622	1.30	.0729404	.10	.0439	.0079	$\sigma(.12) = .27$
	$b = 1.0$.443985	1.21	.0793518	.10	.04838	.00868	$\sigma(.125) = .035$
	$b = 10$.44334	1.20	.080233	.10	.0491	.0088	$\sigma(.130) = .0035$
$a = \frac{1}{20}$	$b = 1$							
	$k = 0$.1471706	1.20		.065	.00886		
	$k = .695$.32936	1.02	.10005136	.04	.0607	.0167	$\sigma(.125) = .016$

Table. The first group bshows the effect of increasing the elementary charge, $k = e^2/G$. The second group shows that a change in the strength b of the free energy affects only the density σ . The last entry give the highest value of k for which a solution was found.

5. Discussion

5.1. The photon gas

There seems to have been no previous discussion of the role of the photon gas in a non neutral plasma. The need for an action principle is very keenly felt. The inspiration for the specific action (3.1) is the form of the action (1.1) that has been successful in the treatment of the matter component. The appropriateness of treating the global electromagnetic field as a complement of the photon gas (the inclusion of the constant term in the coefficient of F^2 in the action) is a guess that must be evaluated *a posteriori*.

Our initial choice for the functional $W[\sigma]$,

$$W[\sigma] = b\sigma^2, \quad b \text{ constant},$$

is motivated within the context by simplicity and asymptotics. As has been stressed elsewhere, our program is to study the coupling of the gravitational field to interesting field theoretic models via an action principle. The physical interpretation has to come from the model itself and a confrontation with classical thermodynamics is not guaranteed to give results that are entirely as expected, especially in the case of strong fields [F2]. But a difficulty arises when it comes to choosing the detailed form of the action. In this paper we are testing an action of the form (3.1), parameterized by the choice of $W[\sigma]$, for a particular choice of this functional, and it is interesting to ask what are the thermodynamic implications. Bluntly: what choice of $W[\sigma]$ is favored by classical thermodynamics?

As was already pointed out, the choice $W[\sigma] = b\sigma^2$ gives the familiar relation between the energy density and the pressure of a pure photon gas. The association of σ with F^2 , Eq.(3.7), allows us to interpret the on shell value of the potential as being proportional to F^4 , a Born-Infeld modifier that can be attributed to the scattering of light by light, which determines the numerical value of the constant b . (Since b is a very large, the photon density will normally be very small.) This photon self-interaction is the physical origin of the photon pressure, allowing the photon gas to transmit sound at velocity $c/\sqrt{3}$.

The inclusion of charge, and the need for an electrostatic field of the form q/r , forces us to include the term $-F^2$ in the action density. An implication of this is that the ratio (total energy density)/(total pressure) is now longer equal to 3 in the presence of charged matter. If this conclusion is correct then we should not expect this last relation to hold except in the case of a pure photon gas, with no matter present, whether charged or not.

Each equation of state, for matter and for the photon gas, can be characterized as isentropic. If the matter component behaves as an ideal gas, then the temperature is proportional to p/ρ and thus to the Emden function, $f \propto T$. Asymptotically, f falls off as $1/r$; hence $T \propto 1/r$. If the photon gas behaves as expected by the thermodynamic interpretation, then the energy density, in the absence of matter, should be proportional to the fourth power of the temperature, hence $\sigma \propto T^2$. In a star made up of charged matter and photons, both σ and the photon energy density fall off as $1/r^4$, and this component of the energy density is not 3 times the corresponding component of the pressure.

Strict thermodynamic equilibrium would require that both temperatures be the same; hence σ should fall off as $1/r^4$, as predicted by Eq.(3.7). The model is thus internally consistent with a thermodynamic equilibrium at large distances in the sense that the two temperatures are at least proportional and perhaps, with some fine tuning, equal. Near the center things are very different, for the Emden function is approximately flat and non zero, while σ tends to zero. However, the naive expectation that each component of the mixture behaves as if it were alone (and in the absence of gravity) was already discredited in the preceding paragraph. Probably one should define equilibrium in mechanical terms, as a static solution, and attempt to find a good definition of temperature for the mixture.

The most spectacular result of the numerical calculations, with $W[\sigma] \propto \sigma^2$, is the fact that the density σ of the photon gas turns out to have a profile that is very different from that of the matter density. This is not consistent with the practice of postulating that the two pressure profiles are proportional to each other. (The traditional approach does not easily accomodate difference in pressure.) In the case of a non neutral plasma they seem to be greatly different.

Lastly, it may be pointed out that the traditional approach also falls short of incorporating all the expectations based on thermodynamics, even in the case of a neutral gas. It is usual to assume that the pressures of radiation and of matter are proportional. For a perfect gas it is $\rho = f^{n+1} \propto T^{n+1}$, so this comes out right only in a region where $n = 3$.

2. Suggestions

The action (3.1), originally inspired by analogy with the matter action (1.1) has turned out to incorporate some if not all of our experience in dealing with electromagnetic phenomena. Our attempts to justify the specific form of this action has led to a partial understanding of the interpretation of the density σ . Indeed, it can be seen as an effective representative of the dielectric properties of the photon gas. In the case studied here, in the absence of flow and of magnetic fields,

$$\sigma F^2 = -\sigma \vec{E}^2 = -\vec{D} \cdot \vec{E}.$$

Interactions between the electromagnetic field and charged matter may be adequately described by the coupling to the conserved current, but it seems plausible that dielectric

properties of the matter gas may have to be taken into account separately. In the case of neutral matter this becomes dominant and it may be thought that the main source of matter-field interaction may be the inclusion of a dielectric modification of Maxwell's action:

$$- \int dx \sqrt{-g} (1 + \epsilon) F^2,$$

where the field ϵ is plausibly taken to be proportional to the density ρ .

The reality of the photon gas strongly suggests that there may be circumstances in which an analogous graviton gas may become interesting. Indeed, this suggestion has been made in the context of stars - gravistars [MM] - as well as an effect to be taken into account in cosmology [R].

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